Role of Part-Inversion in Fluid-Structure Problems with Mixed Variables

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REFERENCE 1 describes with examples a promising formulation for a linear elastic structure coupled to heavy compressible inviscid fluid;

$$\begin{bmatrix} -M & T \\ T^T & K_F \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ p \end{Bmatrix} = \begin{bmatrix} K & O \\ O & M_F \end{bmatrix} \begin{Bmatrix} u \\ -\ddot{p} \end{Bmatrix} \quad (1)$$

Here $\ddot{u} = d^2u/dt^2$ is a vector of nodal accelerations, M and K are the mass and stiffness matrices for the structure, and u gives its nodal displacements. p gives the pressures at the nodes within the fluid and on the fluid-structure interface, and M_F and K_F are matrices such that the fluid behaviour alone can be represented by $M_F\ddot{p} + K_Fp = 0$. Thus M_F and K_F bear only a formal resemblance to M and K. T is a rectangular coupling matrix having a role as in Pian's early mixed-variable formulations.[2]

Equations (1) are also mixed in the unfortunate sense that time derivatives occur on both sides. This may be remedied by a partitioned solution which leads to natural frequency calculations[1] but this is somewhat cumbersome. However, the algebra by which the process was derived suggests a numerical technique which is elegant, instructive, and successful.

Our goal is to convert Eq. (1) to the normal form for a set of second-order, ordinary differential equations

$$[Q]\ \{\ddot{r}\} = \{r\}$$

Consider first the structural case $Mu = \lambda Ku$. Both M and K are represented in core as triangular matrices, being symmetric.[3] A solution procedure for solving this problem with an economical use of computer core storage has been suggested by Torson.[4] If K is positive-definite it can be over-written by its Cholesky square root L, a lower triangular matrix, where $K = LU$, $U = L^T$. The problem now reduces to one involving a single matrix, $Qv = \lambda v$ where $v = Uu$ and $Q = L^{-1}MU^{-1}$. The transformation to Q is achieved in two steps. First L is over-written by L^{-1}. Then M can be over-written by the triple product Q, starting at the final complete row and using an additional vector for temporary storage.[4] With Q and L^{-1} on core, v and λ are found by the Householder or power technique with little extra storage, and hence $u = U^{-1}v$.

The same approach extends to Eq. (1), which can be regarded symbolically as $M^*y = K^*z$. This transforms into $Q^*\{U^*y\} = \{U^*z\}$ which expands to

$$Q^* \begin{Bmatrix} U\ddot{u} \\ U_Fp \end{Bmatrix} = \begin{Bmatrix} Uu \\ -U_F\ddot{p} \end{Bmatrix} \quad (2)$$

where $M_F = L_FU_F$, $U_F = L_F^T$. Now Q^* may be further transformed by Asplund's part-inversion scheme[5]

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{Bmatrix} \ddot{v} \\ q \end{Bmatrix} = \begin{Bmatrix} v \\ -\ddot{q} \end{Bmatrix}$$

where $v = Uu$ and $q = U_Fp$

$$\therefore \begin{bmatrix} Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}, & -Q_{12}Q_{22}^{-1} \\ -Q_{22}^{-1}Q_{21}, & -Q_{22}^{-1} \end{bmatrix} \begin{Bmatrix} \ddot{v} \\ \ddot{q} \end{Bmatrix} = \begin{Bmatrix} v \\ q \end{Bmatrix} \quad (3)$$

This is easily achieved in practice by interchanging one pair of variables, q_i and $\ddot{q}_i$, at a time, with no extra storage requirements. In essence this is the gaussian reduction process starting from the last row and working upwards, eliminating one variable at a time and rearranging the corresponding equation. Indeed, with N structural and n fluid degrees of freedom, the storage requirements can be reduced to $N^2 + Nn + n^2 + 2(N + n)$ words, apart from any special requirements of the eigenvalue routine. With the algorithms just described, the formulation therefore appears highly competitive.

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Buckling of Truncated Conical Shells under Axial Compression

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Introduction

THE elastic stability of truncated conical shells subjected to axial compression has been studied by several researchers.[1–4] In these studies, however, only approximate solutions are obtained by ignoring some of the boundary conditions. Recently, accurate solutions of this problem were obtained by Singer et al.[5] for simply supported shells, but clamped shells have not been treated.‡

In this paper, applying the same method as used for the corresponding problem under torsion,[6] the problem is accurately analysed under four sets of boundary conditions, including both clamped and simply supported cases. Through detailed calculations, the correlations to the buckling of equivalent cylindrical shells are clarified, which facilitates the estimation of critical load for any given conical shell. The results here obtained for simply supported cases are ascertained to be in good agreement with those obtained by Singer et al.[5]

Analysis

Assume that a truncated conical shell is subjected to the axial load P applied along the edges as shown in Fig. 1. Assuming the momentless state of stress for the prebuckling deformation, the Donnell type basic equations governing the critical state of the shell and the relations among the stress function, membrane forces, and displacements may be given in nondimensional form by[7]

$$\nabla^4 f + x^{-1}w_{,xx} = 0 \quad (1)$$

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‡ After the completion of the present study, the authors were informed of the Singer paper through private communication.